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LETTER TO THE EDITOR

Generalized q-fermion oscillators and q-coherent states

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Abstract. The algebra of q-fermion operators, developed earlier by two of the present authors is re-examined. It is shown that these operators represent particles that are distinct from usual spacetime fermions except in the limit $q=1$. It is shown that it is possible to introduce generalized q-oscillators defined for $-\infty<q\leq 1$. In the range $-\infty<q<0$, these coincide with the q-boson operators and for $0<q\leq 1$ they coincide with q-fermions. The ordinary bosons and fermions may be identified with the limits $q=-1$ and $+1$ respectively. Generalized q-fermion coherent states are constructed by utilizing a nonlinear shift automorphism of the algebra of q-fermion operators. These are compared with the coherent states defined as eigenstates of annihilation operator. Matrix elements of the shift operator in the Fock space basis are evaluated.

Much attention has been focused recently on the algebra of q-boson oscillators [1, 2]. It has been conjectured that these oscillators perhaps play the role of the ordinary harmonic oscillators at Planck length scale [2]. The q-boson operator algebra has been used as a tool for constructing the highest weight representations of $\mathrm{SU}_q(2)$ [1, 2] and other quantum groups. q-boson coherent states have been studied in [3, 4]. These coherent states show interesting squeezing properties.

It is natural to investigate the properties of q-fermion oscillators. Two of the present authors [5] proposed an algebra of q-fermion creation and annihilation operators of the form

$$f_q f_q^\dagger + \sqrt{q}\, f_q^\dagger f_q = q^{-N_f/2} \tag{1}$$

where the number operator $N_f(\neq f_q^\dagger f_q)$ satisfies

$$[N_f, f_q^\dagger] = f_q^\dagger \qquad [N, f_q] = -f_q. \tag{2}$$

It was shown in [5] that for $0<q<1$ any number of q-fermions can occupy a given state in contrast to the case of ordinary fermions. It was further shown that when $q=1$, the nilpotency relations $f_{q=1}^2=0, f_{q=1}^{\dagger 2}=0$ are realized in the weak sense; i.e. $f^2|n\rangle_F=0$; $(f^\dagger)^2|n\rangle_F=0$, where $|n\rangle_F$ spans the fermion Fock space, thereby reducing to the usual fermion operators when $q=1$. The above algebra of q-fermions was used in [5] to construct the q-superalgebra of the supersymmetric oscillator and its irreducible representations. Recently, q-fermion operator algebra given by (1) and (2) has been derived by a Wigner–Inönu type contraction of the superalgebra Osp(1/2) [6]. It can also be derived from the $\mathrm{SU}_{p,q}(2)$ algebra [7] by taking the limit $p=-q$. There is an alternate version of q-fermions which can be obtained from $\mathrm{Sl}_q(1/1)$ [6] by contraction. This results in the following anticommutation relations.

$$a_q a_q^\dagger + q a_q^\dagger a_q = q^{+N} \tag{3}$$

$$a_q^2 = a_q^{\dagger 2} = 0 \qquad [N, a_q^\dagger] = a_q^\dagger \qquad [N, a_q] = -a_q. \tag{4}$$

Note the sign of the exponent in q^N. By making the transformation [8]

$$a = q^{-N/2} a_q \qquad a^\dagger = a_q^\dagger q^{-N/2} \tag{5}$$

the equations (3) and (4) reduce to

$$aa^\dagger + a^\dagger a = 1 \qquad a^2 = a^{\dagger 2} = 0 \qquad [N, a^\dagger] = a^\dagger \qquad [N, a] = -a. \tag{6}$$

The algebra of (6) of operators a, $a^\dagger$, and N is the Heisenberg algebra of ordinary fermions and thus equations (3) and (4) do not constitute a generalization. In view of the fact that a number of authors have discussed properties of q-oscillators based on (3) and (4), it is important to point out that the generalization via equations (1) and (2) is not only non-trivial but provides at the same time a scheme for unifying q-boson and fermion oscillators in terms of generalized q-oscillators.

Equations (1) and (2) allow, for $0 < q < 1$, any number of q-fermions in a given state. It therefore seems reasonable to expect no fundamental difference between the q-fermions and q-bosons. To see the connection between these operators, let us define in equations (1) and (2) the following transformations:

$$f_q = q^{-N/4} F \qquad f_q^\dagger = F^\dagger q^{-N/4} \tag{7}$$

(we have dropped the suffix on the number operator N). The basic anticommutation relations now become

$$FF^\dagger + qF^\dagger F = 1 \tag{8}$$

$$[N, F^\dagger] = F^\dagger \qquad [N, F] = -F. \tag{9}$$

Equivalently one may introduce the number operator N by

$$FF^\dagger - F^\dagger F = (-q)^N. \tag{10}$$

We shall show, below, that q in (8) must be taken to be real and in the range $0 < q < 1$, in order to be able to construct a Fock space based on the vacuum $|0\rangle_F$; $F|0\rangle_F = 0$ for these oscillators. Iterating (8), we arrive at the formula

$$F(F^\dagger)^n - (-q)^n (F^\dagger)^n F = [n; q]_F (F^\dagger)^{n-1} \tag{11}$$

where

$$[n; q]_F = [1 - (-q)^n]/(1+q). \tag{12}$$

Defining the vacuum state $|0\rangle_F$ by $F|0\rangle_F = 0$; we can construct normalized n q-fermions state by

$$|n; q\rangle_F = \frac{1}{\sqrt{[n; q]_F!}} (F^\dagger)^n |0\rangle \tag{13}$$

where

$$[n; q]_F! = [n; q]_F [n-1; q]_F \ldots [2; q]_F [1; q]_F. \tag{14}$$

It is readily checked that

$$F|n; q\rangle_F = \sqrt{[n; q]_F}\, |n-1; q\rangle_F \tag{15}$$

$$F^\dagger |n; q\rangle_F = \sqrt{[n+1; q_F}\, |n+1; q\rangle_F. \tag{16}$$

It can be seen from the definition (11) of $[n; q]_F$ that for $q > 1$ $[n; q]_F$ is negative for even values of n, in which case the norm of the Fock space vectors $|n; q\rangle_F$ is ill-defined.

Thus we take $0<q\leqslant 1$. Furthermore, it can be seen from (14) and (11) that when $q\to 1 (F^\dagger)^n|0\rangle_F=\sqrt{[n;q]_F!}|n;q\rangle_F=0$ for $n>1$. On the other hand for $q\neq 1$, it is easy to see from the iteration formula (10) that imposing $(F^\dagger)^n=0$ for $n>1$ leads to inconsistencies.

Recall that the q-boson operators A, $A^\dagger$ satisfy the following commutation relations

$$A_qA_q^\dagger-qA_q^\dagger A_q=1 \tag{17}$$

and

$$[N_B, A_q^\dagger]=A_q^\dagger \qquad [N_B, A_q]=-A_q$$

where q in (17) is taken to be positive and $0<q<\infty$. N_B, in the above, is the boson number operator. We formally extend relations (8) and (9) to negative values of q. For $-\infty<q<0$, change q to $-q$ and redefine $F_{-q}=A_q$. We then obtain the commutation relations (17) with $q>0$. Thus, we may define generalized q-oscillators by (8) and (9) in the interval $-\infty<q\leqslant 1$ such that they represent q-bosons for $-\infty<q<0$, while they represent q-fermions for $0<q\leqslant 1$.

We now construct q-fermion coherent states (qFCS). Klauder and Skagerstam [9] define, for the ordinary ($q=1$) fermions, coherent state $|\psi\rangle$ as

$$|\psi\rangle=\exp[\psi a^\dagger-a\psi^\dagger]|0\rangle \tag{18}$$

where ψ and $\psi^\dagger$ are anticommuting Grassmann variables (i.e. $\psi\psi^\dagger+\psi^\dagger\psi=0$; $\psi^2=0=\psi^{\dagger 2}$). This construction is based on the existence of shift automorphism of the algebra of creation and destruction operators. Denoting by $D(\psi)$

$$D(\psi)=\exp(\psi a^\dagger-a\psi^\dagger) \tag{19}$$

we find

$$D(\psi)aD^\dagger(\psi)=a-\psi. \tag{20}$$

For qFCS it is clear from (1) or (8) that simple shift does not preserve the basic anticommutation relation. Zhedanov [10] has recently constructed a nonlinear shift automorphism for q-bosons. This method can be adapted for q-fermions. We shall construct qFCS using pseudo-Grassmann variables, which in the limit $q=1$ reduces to ordinary fermion coherent states. Let us define nonlinear shift operators $C(\psi)$ and $C^\dagger(\psi)$ by

$$C(\psi)=F[1-(-q)^Nw]^{1/2}-\psi q^N \tag{21}$$

$$C^\dagger(\psi)=[1-(-q)^Nw]^{1/2}F^\dagger-\psi^\dagger q^N \tag{22}$$

where w is a c-number, to be determined by requiring

$$C(\psi)C^\dagger(\psi)+qC^\dagger(\psi)C(\psi)=1. \tag{23}$$

In (22), ψ, $\psi^\dagger$ are taken as pseudo-Grassmann variables. By this we mean that $\psi\psi^\dagger+\psi^\dagger\psi=0$; but ψ and $\psi^\dagger$ are not nilpotent; $\psi^n\neq 0$, $(\psi^\dagger)^n\neq 0$. Further ψ and $\psi^\dagger$ are taken to anticommute with F and $F^\dagger$; but commute with the number operator N.

A straightforward computation shows that $C(\psi)$ and $C^\dagger(\psi)$ define an automorphism of the algebra (8); that is they satisfy equation (23), provided we take

$$w=\psi^\dagger\psi(1-q)/q. \tag{24}$$

Note that when $q\to 1$, $C(\psi)\to F-\psi$ and $C^\dagger(\psi)\to F^\dagger-\psi^\dagger$; reproducing the automorphism of the algebra of ordinary fermions.

There should thus exist a unitary operator $U(\psi)$ that maps F into $C(\psi)$:

$$U(\psi)FU^\dagger(\psi)=C(\psi) \tag{25}$$

$$U(\psi)F^\dagger U^\dagger(\psi)=C^\dagger(\psi). \tag{26}$$

We now define qFCS by

$$|\tilde{\psi}\rangle_q=U(\psi)|0\rangle_q \qquad C(\psi)|\tilde{\psi}\rangle_q=0. \tag{27}$$

Now, from definitions (21) and (22) we have

$$C(-\psi)|0\rangle_q=\psi|0\rangle_q \tag{28}$$

then $|0\rangle_q$ may be interpreted as a q-coherent state of $C(-\psi)$. Substituting (25) in (28) we find

$$FU^\dagger(-\psi)|0\rangle_q=\psi U^\dagger(-\psi)|0\rangle_q. \tag{29}$$

Thus the state $U^\dagger(-\psi)|0\rangle_q$ is a q-coherent state in the sense of being the eigenstate of the annihilation operator F. Note that $U^\dagger(-\psi)\neq U(\psi)$, and hence the coherent state $U^\dagger(-\psi)|0\rangle_q$ is not identical to the coherent state $|\tilde{\psi}\rangle_q$. Let us define $|\psi\rangle_q=U^\dagger(-\psi)|0\rangle_q$. It is, in fact possible to construct the state $|\psi\rangle_q$ directly, by following the procedure used for q-bosons. We shall show that the normalized state $|\psi\rangle_q$ is given by

$$|\psi\rangle_q=(\exp_q(\psi^\dagger\psi)^{-1/2}\exp_q(-\psi F^\dagger)|0\rangle_q \tag{30}$$

where

$$\exp_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{[n;q]_F!}. \tag{31}$$

$[n;q]_F!$ is defined in (12). The q-exponential defined in [31) is seen to be uniformly and absolutely convergent for $q<1$ with a radius of convergence $R=1/(1+q)$. It is possible to re-define a q-exponential function, $e_q(x)$, that is convergent for $q<1$ for all values of x. Using the anticommuting properties of ψ, it is easily shown that

$$F|\psi\rangle_q=\psi|\psi\rangle_q. \tag{32}$$

Finally one can determine the matrix elements of $U^\dagger(\psi)$ in the Fock space. We derive a recursion formula by calculating the matrix element $\langle m|(-q)^N U^\dagger(\psi)|n\rangle$ in two ways. Now

$$\langle m|(-q)^N U^\dagger(\psi)|n\rangle=(-q)^m U^\dagger_{m,n}\equiv(-1)^m x(m)\,U^\dagger_{m,n} \tag{33}$$

where

$$x(m)=e^{-\omega m} \qquad q\equiv e^{-\omega}(\omega>0). \tag{34}$$

On the other hand from equations (10) and (25), we have

$$\begin{aligned}\langle m|(-q)^N U^\dagger(\psi)|n\rangle&=\langle m|U^\dagger(\psi)U(\psi)(-q)^N U^\dagger(\psi)|n\rangle\\&=\langle m|U^\dagger(\psi)(CC^\dagger-C^\dagger C)|n\rangle\\&=g_n U^\dagger_{m,n}+d_n U^\dagger_{m,n-1}\psi^\dagger+d_{n+1}U^\dagger_{n,n+1}\psi\end{aligned} \tag{35}$$

where

$$\begin{aligned}g_n&=(-q)^n 1+w-(-q)^n\frac{w(1+q^2)}{1-q}\\d_n&=(-q)^{n-1}[(1-w(-q)^n)(1-(-q)^n)(1+q)]^{1/2}.\end{aligned} \tag{36}$$

In arriving at (35), we have made use of the following properties of ψ: we take ψ to commute with $|0\rangle_F$

$$\psi|0\rangle_F = |0\rangle_F\psi \tag{37}$$

then

$$\psi(F^\dagger)^n|0\rangle_F = (-1)^n(F^\dagger)^n|0\rangle_F\psi. \tag{38}$$

It is seen from (33) and (35) that one can express $U^\dagger_{m,n}(\psi)\psi^n$ in the form

$$U^\dagger_{m,n}(\psi)\psi^n = \langle m|U^\dagger(\psi)|0\rangle C_n(x(m)) \tag{39}$$

where $C_n(x(m))$ is a system of orthogonal polynomials of the discrete argument $x(m) \equiv \exp(-\omega m)$. The orthogonality properties of $C_n(x(m))$ arise as a consequence of unitarity of $U^\dagger(\psi)$.

$$\sum_{m=0}^{\infty} W_m C_n(x(m)) C_{n'}(x(m)) = (\psi^\dagger\psi)^n \delta_{n,n'}. \tag{40}$$

The weight function W_m for these polynomials has the form

$$\begin{aligned} W_m &= |\langle m|U^\dagger(\psi)|0\rangle|^2 \\ &= |\langle m|-\psi\rangle_q|^2 \\ &= \frac{1}{(\exp_q|\psi|^2)} \frac{(\psi^\dagger\psi)^m}{[m;q]!}. \end{aligned} \tag{41}$$

These polynomials are not identical to the ones for q-bosons. It will be of interest to investigate in detail the properties of these polynomials.

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